

THE LINEAR BOUND FOR HAAR MULTIPLIER PARAPRODUCTS

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ABSTRACT. We study the natural resolution of the conjugated Haar multiplier T_σ :

$$M_{w^{\frac{1}{2}}} T_\sigma M_{w^{-\frac{1}{2}}} = \left(\widehat{P_{w^{\frac{1}{2}}}^{(0,1)}} + \widehat{P_{w^{\frac{1}{2}}}^{(1,0)}} + \widehat{P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)}} \right) T_\sigma \left(\widehat{P_{w^{-\frac{1}{2}}}^{(0,1)}} + \widehat{P_{w^{-\frac{1}{2}}}^{(1,0)}} + \widehat{P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)}} \right),$$

where each M_φ is decomposed into its canonical paraproduct decomposition. We prove that each constituent operator obtained from this resolution has a linear bound on $L^2(\mathbb{R}^d; w)$ in terms of the A_2 characteristic of w . The main tools used are a “product formula” for Haar coefficients, the Carleson Embedding Theorem, the linear bound for the square function, and the well-known linear bound of T_σ on $L^2(w)$.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let $L^2 \equiv L^2(\mathbb{R}^d)$ denote the space of square integrable functions over \mathbb{R}^d . For a weight w , i.e., a positive locally integrable function on \mathbb{R}^d , we set $L^2(w) \equiv L^2(\mathbb{R}^d; w)$. In particular, we will be interested in A_2 weights, which are defined by:

$$[w]_{A_2} \equiv \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I,$$

where $\langle w \rangle_I$ denotes the average of w over a cube I .

An operator T is bounded on $L^2(w)$ if and only if $M_{w^{\frac{1}{2}}} T M_{w^{-\frac{1}{2}}}$ - the conjugation of T by the multiplication operators $M_{w^{\pm\frac{1}{2}}}$ - is bounded on L^2 . Moreover, the operator norms are equal:

$$\|T\|_{L^2(w) \rightarrow L^2(w)} = \left\| M_{w^{\frac{1}{2}}} T M_{w^{-\frac{1}{2}}} \right\|_{L^2 \rightarrow L^2}.$$

In the case that T is a dyadic operator adapted to a dyadic grid \mathcal{D} , it is natural to study weighted norm properties of T by decomposing the multiplication operators $M_{w^{\pm\frac{1}{2}}}$ into their

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canonical paraproduct decompositions relative to the grid \mathcal{D} , i.e.

$$M_{w^{\pm\frac{1}{2}}} f = \widehat{P_{w^{\pm\frac{1}{2}}}^{(0,1)}} f + \widehat{P_{w^{\pm\frac{1}{2}}}^{(1,0)}} f + P_{\langle w^{\pm\frac{1}{2}} \rangle}^{(0,0)} f$$

(the paraproduct operators are defined in the next section) and then decomposing $M_{w^{\frac{1}{2}}} T M_{w^{-\frac{1}{2}}}$ into the nine canonical individual paraproduct composition operators:

$$\begin{aligned} (1.1) \quad M_{w^{\frac{1}{2}}} T M_{w^{-\frac{1}{2}}} &= \left(\widehat{P_{w^{\frac{1}{2}}}^{(0,1)}} + \widehat{P_{w^{\frac{1}{2}}}^{(1,0)}} + P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} \right) T \left(\widehat{P_{w^{-\frac{1}{2}}}^{(0,1)}} + \widehat{P_{w^{-\frac{1}{2}}}^{(1,0)}} + P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)} \right) \\ &\equiv Q_{T,w}^{(0,1),(0,1)} + Q_{T,w}^{(0,1),(1,0)} + Q_{T,w}^{(0,1),(0,0)} \\ &\quad + Q_{T,w}^{(1,0),(0,1)} + Q_{T,w}^{(1,0),(1,0)} + Q_{T,w}^{(1,0),(0,0)} \\ &\quad + Q_{T,w}^{(0,0),(0,1)} + Q_{T,w}^{(0,0),(1,0)} + Q_{T,w}^{(0,0),(0,0)}, \end{aligned}$$

where $Q_{T,w}^{(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4)}$ is defined in the obvious way from the expression above. If one could show that the operator norms of the $Q_{T,w}^{(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4)}$ are linear in the A_2 characteristic,

$$\left\| Q_{T,w}^{(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4)} \right\|_{L^2 \rightarrow L^2} \lesssim [w]_{A_2}$$

it then becomes reasonable to expect that the canonical decomposition of a dyadic operator T into its paraproduct compositions $Q_{T,w}^{(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4)}$ will inherit the salient properties of T without losing anything of importance. Of course, these dyadic paraproduct compositions can be expected to yield to structured dyadic proof strategies.

This idea has been successfully used to study decompositions of the Hilbert transform in [6]. We now extend this idea to the martingale transforms. Specifically, let $\{\sigma_I\}_{I \in \mathcal{D}}$ denote a sequence of $2^d - 1 \times 2^d - 1$ diagonal matrices indexed by the dyadic cubes with diagonal entries denoted by $(\sigma_I)_{\alpha\alpha} \equiv \sigma_{I,\alpha}$ for $\alpha = 1, \dots, 2^d - 1$. Define

$$T_\sigma f \equiv \sum_{I \in \mathcal{D}} \left(\sigma_I \widehat{f}(I) \right) \cdot h_I \quad \forall f \in L^2,$$

where h_I is the vector of Haar functions adapted to the cube I and $\widehat{f}(I)$ is the vector of Haar coefficients associated to the function f . For precise definitions of these Haar objects, see Section 2 and for a precise definition of T_σ , see the beginning of Section 3. It is well known and simple to see that

$$\|T_\sigma\|_{L^2 \rightarrow L^2} \leq \|\sigma\|_\infty,$$

where

$$\|\sigma\|_\infty \equiv \sup_{I \in \mathcal{D}} \sup_{1 \leq \alpha \leq 2^d - 1} |\sigma_{I,\alpha}|.$$

A similar norm bound holds for $L^2(w)$. Specifically, in [9], J. Wittwer established the following result in one-dimension and using related arguments, D. Chung obtained the d -dimensional analogue in [1].

Theorem 1.1 (Linear Bound for Martingale Transforms).

$$\|T_\sigma\|_{L^2(w) \rightarrow L^2(w)} \lesssim [w]_{A_2} \|\sigma\|_\infty.$$

Wittwer only established the result for the case where each $\sigma_I \in \{\pm 1\}$, but the general case follows using the same arguments. F. Nazarov, S. Treil, and A. Volberg obtained more general results in [3, 4], where they showed that certain testing conditions are sufficient to determine when Haar multipliers and related operators are bounded from $L^2(w)$ to $L^2(v)$.

Here, we study the paraproduct decomposition of T_σ and establish the following result:

Theorem 1.2. *let $\{\sigma_I\}_{I \in \mathcal{D}}$ denote a sequence of $2^d - 1 \times 2^d - 1$ diagonal matrices indexed by the dyadic cubes and let w be an A_2 weight. Then, each paraproduct composition in the canonical resolution of T_σ can be controlled by a linear power of $[w]_{A_2}$, i.e.*

$$\left\| Q_{T_\sigma, w}^{(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4)} \right\|_{L^2 \rightarrow L^2} \lesssim [w]_{A_2} \|\sigma\|_\infty,$$

for each $Q_{T_\sigma, w}^{(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4)}$ in (1.1).

The proof relies heavily on arguments appearing in [6], especially the use of a “product formula” for Haar coefficients, the Carleson Embedding Theorem, and the linear bound for the square function. To handle the final resolvent paraproduct, we must rely on Theorem 1.1. Obtaining the bound independent of Theorem 1.1 is currently an open question.

2. NOTATION AND USEFUL FACTS

Before proving our main result, we collect necessary notation and estimates. Throughout this paper $A \equiv B$ means that the expressions are equal by definition, and $A \lesssim B$ means that there exists a constant c_d , which may depend on the dimension d , such that $A \leq c_d B$.

Let \mathcal{D} denote the usual dyadic grid of cubes in \mathbb{R}^d . For $I \in \mathcal{D}$, let $\mathcal{C}_1(I)$ denote the 2^d children of I . Note that each child $J \in \mathcal{C}_1(I)$ satisfies $|J| = 2^{-d} |I|$. Further, for $d \in \mathbb{N}$, set

$$\Gamma_d \equiv \{0, 1\}^d \setminus \{(1, \dots, 1)\},$$

and fix an enumeration of this set for the rest of the paper. Elements of this set will be denoted by lowercase greek letters.

2.1. Wilson’s Haar System. While we would like to use the standard Haar system in the analysis below, it is more convenient to use an orthonormal system developed by M. Wilson in [8]. To construct it, we need the following lemma. It is worth mentioning that property (i) did not appear in Wilson’s original lemma but was added by D. Chung in [1].

Lemma 2.1 (Wilson, [8, Lemma 2]). *Let $I \in \mathcal{D}$. Then there are $2^d - 1$ pairs of sets $\{(E_{\alpha, I}^1, E_{\alpha, I}^2)\}_{\alpha \in \Gamma_d}$ such that*

- (i) *For each $\alpha \in \Gamma_d$, $|E_{\alpha, I}^1| = |E_{\alpha, I}^2|$;*
- (ii) *For each α and $s = 1, 2$, $E_{\alpha, I}^s$ is a non-empty union of cubes from $\mathcal{C}_1(I)$;*
- (iii) *For each α , $E_{\alpha, I}^1 \cap E_{\alpha, I}^2 = \emptyset$;*
- (iv) *For every $\alpha \neq \beta$ one of the following must hold:*
 - (a) $E_{\alpha, I}^1 \cup E_{\alpha, I}^2$ is entirely contained in either $E_{\beta, I}^1$ or $E_{\beta, I}^2$;
 - (b) $E_{\beta, I}^1 \cup E_{\beta, I}^2$ is entirely contained in either $E_{\alpha, I}^1$ or $E_{\alpha, I}^2$;
 - (c) $(E_{\beta, I}^1 \cup E_{\beta, I}^2) \cap (E_{\alpha, I}^1 \cup E_{\alpha, I}^2) = \emptyset$.

Set $E_{\alpha, I} \equiv E_{\alpha, I}^1 \cup E_{\alpha, I}^2$. It is important to observe that $|E_{\alpha, I}| \approx |I|$ for some dimensional constants. Further, given any $E_{\alpha, I}$ and $E_{\beta, J}$, it follows from the properties of \mathcal{D} and (iv) that one of the following must hold: $E_{\alpha, I} \subsetneq E_{\beta, J}$, $E_{\beta, J} \subsetneq E_{\alpha, I}$, or $E_{\alpha, I} = E_{\beta, J}$. Given this

collection of sets, we can now introduce Wilson's Haar system for $L^2(w)$. Fix $\alpha \in \Gamma_d$, $I \in \mathcal{D}$ and define:

$$(2.1) \quad h_I^{w,\alpha} \equiv \frac{1}{\sqrt{w(E_{\alpha,I})}} \left[\frac{\sqrt{w(E_{\alpha,I}^1)}}{\sqrt{w(E_{\alpha,I}^2)}} 1_{E_{\alpha,I}^2} - \frac{\sqrt{w(E_{\alpha,I}^2)}}{\sqrt{w(E_{\alpha,I}^1)}} 1_{E_{\alpha,I}^1} \right].$$

It is easy to show that $\{h_I^{w,\alpha}\}_{\alpha \in \Gamma_d, I \in \mathcal{D}}$ is an orthonormal system in $L^2(w)$. When the weight $w \equiv 1$, we denote this collection of functions by $\{h_I^\alpha\}_{\alpha \in \Gamma_d, I \in \mathcal{D}}$. Note that each function h_I^α has a fixed sign on each child cube $I' \in \mathcal{C}_1(I)$. Now, for a fixed dyadic cube J , set

$$\begin{aligned} \widehat{f}(J, \alpha) &\equiv \langle f, h_J^\alpha \rangle_{L^2} \quad \forall \alpha \in \Gamma_d, \\ \widehat{f}(J) &\equiv \left(\widehat{f}(J, \alpha) \right)_{\alpha \in \Gamma_d}, \\ h_J &\equiv (h_J^\alpha)_{\alpha \in \Gamma_d}, \\ \Delta_J f &\equiv \widehat{f}(J) \cdot h_J = \sum_{\alpha \in \Gamma_d} \widehat{f}(J, \alpha) h_J^\alpha. \end{aligned}$$

This means that $\widehat{f}(J)$ is the vector of Haar coefficients of the function f , and h_J is the vector of Haar functions. It is easy to see that the set $\{h_I^\alpha\}_{I \in \mathcal{D}, \alpha \in \Gamma_d}$ is an orthonormal basis for L^2 and so, $f = \sum_{I \in \mathcal{D}} \Delta_I f = \sum_{I \in \mathcal{D}} \widehat{f}(I) \cdot h_I$. This implies

$$\|f\|_{L^2}^2 = \sum_{I \in \mathcal{D}} \left| \widehat{f}(I) \right|^2 = \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} \left| \widehat{f}(I, \alpha) \right|^2 = \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \widehat{f}(I, \alpha) \right|^2.$$

Now given a set E , define $h_E^1 \equiv \frac{1_E}{|E|}$, so that the function is L^1 normalized. We also set $\langle f \rangle_E \equiv \langle f, h_E^1 \rangle_{L^2}$. The Wilson Haar system has the standard martingale property that the average of f over each $E_{\alpha,I}$ satisfies

$$\langle f \rangle_{E_{\alpha,I}} = \sum_{J \in \mathcal{D}} \sum_{\beta \in \Gamma_d} \widehat{f}(J, \beta) \left\langle h_J^\beta, h_{E_{\alpha,I}}^1 \right\rangle_{L^2} = \sum_{J: J \supseteq I} \sum_{\beta: E_{J,\beta} \supseteq E_{\alpha,I}} \widehat{f}(J, \beta) \left\langle h_J^\beta, h_{E_{\alpha,I}}^1 \right\rangle_{L^2}.$$

A fundamental tool in our study will be the product formula for Haar expansions in L^2 . A version of this previously appeared in [7]. Specifically, given two functions f and g in L^2 , we can expand them with respect to this Haar basis and formally obtain

$$\begin{aligned} fg &= \left(\sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} \widehat{f}(I, \alpha) h_I^\alpha \right) \times \left(\sum_{J \in \mathcal{D}} \sum_{\beta \in \Gamma_d} \widehat{g}(J, \beta) h_J^\beta \right) \\ &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \langle g \rangle_{E_{\alpha,I}} \widehat{f}(I, \alpha) h_I^\alpha + \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \langle f \rangle_{E_{\alpha,I}} \widehat{g}(I, \alpha) h_I^\alpha + \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \widehat{g}(I, \alpha) \widehat{f}(I, \alpha) h_{E_{\alpha,I}}^1, \end{aligned}$$

where we use the fact that

$$h_I^\alpha h_J^\beta = \left\langle h_J^\beta, h_{E_{\alpha,I}}^1 \right\rangle_{L^2} h_I^\alpha \text{ whenever } E_{\alpha,I} \subsetneq E_{\beta,J}.$$

Although the product formula above does not necessarily make sense for arbitrary $f, g \in L^2$, it is well-defined if f, g are finite linear combinations of Haar functions. Moreover, for $J \in \mathcal{D}$ and $\beta \in \Gamma_d$, we have

$$(2.2) \quad \widehat{fg}(J, \beta) = \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \widehat{f}(I, \alpha) \widehat{g}(I, \alpha) \left\langle h_{E_{\alpha,I}}^1, h_J^\beta \right\rangle_{L^2} + \widehat{f}(J, \beta) \langle g \rangle_{E_{\beta,J}} + \widehat{g}(J, \beta) \langle f \rangle_{E_{\beta,J}}.$$

For finite linear combinations of Haar functions, this is obtained by simply calculating the Haar coefficient corresponding to $\beta \in \Gamma_d$ and $J \in \mathcal{D}$ in formula for the product fg . If f, g are locally in L^2 , we can approximate them on $E_{\beta,J}$ by finite linear combinations of Haar functions and still deduce (2.2). Primarily, we will use version (2.2) of the product formula. However, it should be noted that support conditions in the first term actually imply that

$$\widehat{fg}(J, \beta) = \sum_{I: I \subseteq J} \sum_{\alpha: E_{\alpha,I} \not\subseteq E_{\beta,J}} \widehat{f}(I, \alpha) \widehat{g}(I, \alpha) \left\langle h_{E_{\alpha,I}}^1, h_J^\beta \right\rangle_{L^2} + \widehat{f}(J, \beta) \langle g \rangle_{E_{\beta,J}} + \widehat{g}(J, \beta) \langle f \rangle_{E_{\beta,J}}.$$

Motivated by these product decompositions, we consider the following dyadic operators. They will be of fundamental importance in this paper. Give a sequence of numbers $a = \{a_{I,\alpha}\}_{I \in \mathcal{D}, \alpha \in \Gamma_d}$ indexed by $I \in \mathcal{D}$ and $\alpha \in \Gamma_d$, we define the following paraproduct type operators:

$$\begin{aligned} P_a^{(0,0)} f &\equiv \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} a_{I,\alpha} \widehat{f}(I, \alpha) h_I^\alpha \\ P_a^{(0,1)} f &\equiv \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} a_{I,\alpha} \langle f \rangle_{E_{\alpha,I}} h_I^\alpha \\ P_a^{(1,0)} f &\equiv \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} a_{I,\alpha} \widehat{f}(I, \alpha) h_{E_{\alpha,I}}^1. \end{aligned}$$

It is easy to see that the operator M_g of multiplication by g can formally be written as

$$(2.3) \quad M_g f = P_{\langle g \rangle}^{(0,0)} f + P_{\widehat{g}}^{(0,1)} f + P_{\widehat{g}}^{(1,0)} f$$

where $\langle g \rangle \equiv \{\langle g \rangle_{E_{\alpha,I}}\}_{I \in \mathcal{D}, \alpha \in \Gamma_d}$ and $\widehat{g} \equiv \{\widehat{g}(I, \alpha)\}_{I \in \mathcal{D}, \alpha \in \Gamma_d}$. We will use (2.3) to decompose the operators $M_{w^{\pm \frac{1}{2}}}$.

2.1.1. Disbalanced Haars. At points in our later arguments, we will use disbalanced Haar functions. To do so, we require some additional notation. Fixing a dyadic cube J , a weight w on \mathbb{R}^d , and $\beta \in \Gamma_d$, we set

$$(2.4) \quad C_J(w, \beta) \equiv \sqrt{\frac{\langle w \rangle_{E_{\beta,J}^1} \langle w \rangle_{E_{\beta,J}^2}}{\langle w \rangle_{E_{\beta,J}}}} \quad \text{and} \quad D_J(w, \beta) \equiv \frac{\widehat{w}(J, \beta)}{\langle w \rangle_{E_{\beta,J}}}.$$

Then we have

$$(2.5) \quad h_J^\beta = C_J(w, \beta) h_J^{w,\beta} + D_J(w, \beta) h_{E_{\beta,J}}^1$$

where $\{h_I^{w,\alpha}\}_{I \in \mathcal{D}, \alpha \in \Gamma_d}$ is the $L^2(w)$ orthonormal system defined in (2.1). To see this, we use the two equations

$$\int_J h_J^{w,\beta} w = 0 \quad \text{and} \quad \int_J \left(h_J^{w,\beta} \right)^2 w = 1$$

to solve for $C_J(w, \beta)$ and $D_J(w, \beta)$. The claimed formula for $D_J(w, \beta)$ follows immediately from the condition that $h_J^{w,\beta}$ have integral zero. Using the second condition and the formula for $D_J(w, \beta)$, one can easily prove that

$$C_J(w, \beta) = \sqrt{\frac{\langle w \rangle_{E_{\beta,J}}^2 - |E_{\beta,J}|^{-1} \widehat{w}(J, \beta)^2}{\langle w \rangle_{E_{\beta,J}}}}.$$

Using basic manipulations, and the fact that $|E_{\beta,J}| = 2|E_{\beta,J}^j|$ for $j = 1, 2$, we have

$$\begin{aligned} \langle w \rangle_{E_{\beta,J}}^2 - |E_{\beta,J}|^{-1} \widehat{w}(J, \beta)^2 &= \frac{1}{|E_{\beta,J}|^2} \left((w(E_{\beta,J}^1) + w(E_{\beta,J}^2))^2 - (w(E_{\beta,J}^2) - w(E_{\beta,J}^1))^2 \right) \\ &= \frac{4}{|E_{\beta,J}|^2} w(E_{\beta,J}^1) w(E_{\beta,J}^2) = \langle w \rangle_{E_{\beta,J}^1} \langle w \rangle_{E_{\beta,J}^2}. \end{aligned}$$

This gives the desired formula for $C_J(w, \beta)$. A useful observation is

$$(2.6) \quad C_J(w, \beta)^2 \leq 4 \langle w \rangle_{E_{\beta,J}} \leq 2^{d+1} \langle w \rangle_J,$$

which follows since each $\langle w \rangle_{E_{\beta,J}^j} \leq 2 \langle w \rangle_{E_{\beta,J}}$ and as $E_{\beta,J}$ contains at least two children of J , $\langle w \rangle_{E_{\beta,J}} \leq 2^{d-1} \langle w \rangle_J$.

2.1.2. Carleson Embedding Theorem. A major tool in this paper is the following modification of the standard Carleson Embedding Theorem to the sets $\{E_{\alpha,I}\}_{I \in \mathcal{D}, \alpha \in \Gamma_d}$. It appears in [1, Theorem 4.3]:

Theorem 2.2 (Modified Carleson Embedding Theorem). *Let w be a weight on \mathbb{R}^d and let $\{a_{\alpha,I}\}_{I \in \mathcal{D}, \alpha \in \Gamma_d}$ be a sequence of nonnegative numbers. Then, there is a constant $A > 0$ such that*

$$\frac{1}{|E_{\alpha,I}|} \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} a_{\beta,J} \langle w \rangle_{E_{\beta,J}}^2 \leq A \langle w \rangle_{E_{\alpha,I}} \quad \forall I \in \mathcal{D}, \alpha \in \Gamma_d,$$

if and only if

$$\sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} a_{\alpha,I} \left\langle w^{\frac{1}{2}} f \right\rangle_{E_{\alpha,I}}^2 \lesssim A \|f\|_{L^2}^2 \quad \forall f \in L^2.$$

2.2. Square Function Estimates.

2.2.1. Square Function Bound. Define the dyadic square function S on L^2 by

$$Sf(x)^2 = \sum_{I \in \mathcal{D}} |\widehat{f}(I)|^2 h_I^1(x) = \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} |\widehat{f}(I, \alpha)|^2 h_I^1(x).$$

It is clear from the definition that $\|Sf\|_{L^2} = \|f\|_{L^2}$. Versions of the square function have been studied in the weighted setting $L^2(w)$ and it has been shown that a linear bound in terms of the A_2 characteristic holds. We point the interested readers to [2].

For our needs, we require a slightly different formulation and so provide an alternate proof of this fact. Using the arguments from Petermichl and Pott [5], we prove

Theorem 2.3. *Let w be an A_2 weight in \mathbb{R}^d . Then*

$$\|Sf\|_{L^2(w)} \lesssim [w]_{A_2} \|f\|_{L^2(w)} \quad \forall f \in L^2(w).$$

Proof. As in [5], without loss of generality, we can assume w and w^{-1} are bounded so long as the bounds do not appear in our final estimates. We first prove the lower bound:

$$(2.7) \quad \|f\|_{L^2(w)}^2 \lesssim [w]_{A_2} \|Sf\|_{L^2(w)}^2 \quad \forall f \in L^2(w).$$

To this end, define the discrete multiplication operator $D_w : L^2 \rightarrow L^2$ by

$$D_w : h_I^\alpha \mapsto \langle w \rangle_I h_I^\alpha \quad \forall I \in \mathcal{D}, \alpha \in \Gamma_d$$

and let M_w denote multiplication by w . Then, we can rewrite (2.7) as:

$$(2.8) \quad \langle M_w f, f \rangle_{L^2} \lesssim [w]_{A_2} \langle D_w f, f \rangle_{L^2}, \quad \forall f \in L^2.$$

First, since w and w^{-1} are bounded, D_w and M_w are bounded and invertible with $M_w^{-1} = M_{w^{-1}}$ and D_w^{-1} defined by

$$D_w^{-1} : h_I^\alpha \mapsto \langle w \rangle_I^{-1} h_I^\alpha \quad \forall I \in \mathcal{D}, \alpha \in \Gamma_d.$$

As in [5], one can convert (2.8) to the equivalent inverse inequality:

$$(2.9) \quad \langle D_w^{-1} f, f \rangle_{L^2} \lesssim [w]_{A_2} \langle M_w^{-1} f, f \rangle_{L^2}, \quad \forall f \in L^2.$$

So, we need to establish:

$$(2.10) \quad \sum_{I \in \mathcal{D}} \langle w \rangle_I^{-1} \left| \widehat{f}(I) \right|^2 \lesssim [w]_{A_2} \|f\|_{L^2(w^{-1})}^2 \quad \forall f \in L^2.$$

As in [5], our first step is to rewrite the sums using disbalanced Haar functions adapted to w using (2.4). To do so, fix a cube J and $\alpha \in \Gamma_d$ and recall that

$$C_J(w, \alpha) \equiv \sqrt{\frac{\langle w \rangle_{E_{\alpha, J}^1} \langle w \rangle_{E_{\alpha, J}^2}}{\langle w \rangle_{E_{\alpha, J}}}} \quad \text{and} \quad D_J(w, \alpha) \equiv \frac{\widehat{w}(J, \alpha)}{\langle w \rangle_{E_{\alpha, J}}}.$$

Then we have

$$h_J^\alpha = C_J(w, \alpha) h_J^{w, \alpha} + D_J(w, \alpha) h_{E_{\alpha, J}}^1,$$

where $\{h_J^{w, \alpha}\}_{J \in \mathcal{D}, \alpha \in \Gamma_d}$ is the previously-defined $L^2(w)$ orthonormal system. Returning to the sum in question, we use the disbalanced Haar functions to write:

$$\begin{aligned} \sum_{I \in \mathcal{D}} \langle w \rangle_I^{-1} \left| \widehat{f}(I) \right|^2 &= \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} \langle w \rangle_I^{-1} \langle f, h_I^\alpha \rangle_{L^2}^2 \\ &= \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} C_I(w, \alpha)^2 \langle w \rangle_I^{-1} \langle f, h_I^{w, \alpha} \rangle_{L^2}^2 \\ &\quad + 2 \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} C_I(w, \alpha) D_I(w, \alpha) \langle w \rangle_I^{-1} \langle f, h_I^{w, \alpha} \rangle_{L^2} \langle f \rangle_{E_{\alpha, I}} \\ &\quad + \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} D_I(w, \alpha)^2 \langle w \rangle_I^{-1} \langle f \rangle_{E_{\alpha, I}}^2 \\ &= S_1 + S_2 + S_3. \end{aligned}$$

Since by (2.6), each $C_I(w, \alpha)^2 \lesssim \langle w \rangle_I$, we can conclude that each $C_I(w, \alpha)^2 \langle w \rangle_I^{-1} \lesssim 1$. This means

$$S_1 \lesssim \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} |\langle f, h_I^{w, \alpha} \rangle_{L^2}|^2 = \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} |\langle w^{-1} f, h_I^{w, \alpha} \rangle_{L^2(w)}|^2 \leq \|f\|_{L^2(w^{-1})}^2.$$

Observe that

$$S_2 \lesssim \left(\sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} C_I(w, \alpha)^2 \langle w \rangle_I^{-1} |\langle f, h_I^{w, \alpha} \rangle_{L^2}|^2 \right)^{\frac{1}{2}} \left(\sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} D_I(w, \alpha)^2 \langle w \rangle_I^{-1} \langle f \rangle_{E_{\alpha, I}}^2 \right)^{\frac{1}{2}}.$$

The first part of the product is the square root of S_1 and the second part is the square root of S_3 . Thus, the proof is reduced to controlling S_3 . We use the modified Carleson Embedding

Theorem. To apply it, we need

$$\begin{aligned} \frac{1}{|E_{\alpha,I}|} \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} D_J(w, \beta)^2 \langle w \rangle_J^{-1} \langle w \rangle_{E_{\beta,J}}^2 &= \frac{1}{|E_{\alpha,I}|} \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \frac{\widehat{w}(J, \beta)^2}{\langle w \rangle_{E_{\beta,J}}^2} \langle w \rangle_J^{-1} \langle w \rangle_{E_{\beta,J}}^2 \\ &\lesssim \frac{1}{|E_{\alpha,I}|} \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \widehat{w}(J, \beta)^2 \langle w \rangle_{E_{\beta,J}}^{-1} \\ &\lesssim [w]_{A_2} \langle w \rangle_{E_{\alpha,I}}, \end{aligned}$$

where the last inequality appears in [1, Proposition 4.9, (4.17)]. Then, the modified Carleson Embedding Theorem implies

$$S_3 = \sum_{I \in \mathcal{D}} \sum_{\alpha \in \Gamma_d} D_I(w, \alpha)^2 \langle w \rangle_I^{-1} \left\langle f w^{\frac{1}{2}} w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}}^2 \lesssim [w]_{A_2} \|f w^{-\frac{1}{2}}\|_{L^2}^2 = [w]_{A_2} \|f\|_{L^2(w^{-1})}^2,$$

which proves the lower square function bound. Given (2.7) for every A_2 weight, the upper square function bound follows almost immediately. Now, for w with w, w^{-1} bounded, the desired inequality is equivalent to

$$\langle D_w f, f \rangle_{L^2} \lesssim [w]_{A_2}^2 \langle M_w f, f \rangle_{L^2}, \quad \forall f \in L^2.$$

To prove that, we require the following operator inequality

$$D_w \leq [w]_{A_2} (D_{w^{-1}})^{-1}.$$

This is immediate since the trivial inequality $\langle w \rangle_I \leq [w]_{A_2} \langle w^{-1} \rangle_I^{-1}$ implies

$$\langle D_w f, f \rangle_{L^2} = \sum_{I \in \mathcal{D}} \langle w \rangle_I \left| \widehat{f}(I) \right|^2 \leq [w]_{A_2} \sum_{I \in \mathcal{D}} \langle w^{-1} \rangle_I^{-1} \left| \widehat{f}(I) \right|^2 = [w]_{A_2} \langle (D_{w^{-1}})^{-1} f, f \rangle_{L^2}.$$

Combining that estimate with (2.9) applied to w^{-1} gives:

$$\langle D_w f, f \rangle_{L^2} \leq [w]_{A_2} \langle (D_{w^{-1}})^{-1} f, f \rangle_{L^2} \lesssim [w]_{A_2}^2 \langle (M_{w^{-1}})^{-1} f, f \rangle_{L^2} = [w]_{A_2}^2 \langle M_w f, f \rangle_{L^2}, \quad \forall f \in L^2,$$

which completes the proof. ■

2.2.2. Key Estimates Deduced from the Square Function. As we have shown above, for $w \in A_2$,

$$\|Sf\|_{L^2(w)}^2 \lesssim [w]_{A_2}^2 \|f\|_{L^2(w)}^2.$$

Applying this inequality to $f = w^{-\frac{1}{2}} \mathbf{1}_{E_{\alpha,I}}$ for $I \in \mathcal{D}, \alpha \in \Gamma_d$ and using some trivial estimates yields the following:

$$(2.11) \quad \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \left| \widehat{w^{-\frac{1}{2}}}(J, \beta) \right|^2 \langle w \rangle_J \lesssim [w]_{A_2}^2 |E_{\alpha,I}| \quad \forall I \in \mathcal{D}, \alpha \in \Gamma_d.$$

A trivial consequence of (2.11) is the following:

$$(2.12) \quad \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \left| \widehat{w^{-\frac{1}{2}}}(J, \beta) \right|^2 \left\langle w^{\frac{1}{2}} \right\rangle_{E_{\beta,J}}^2 \lesssim [w]_{A_2}^2 |E_{\alpha,I}| \quad \forall I \in \mathcal{D}, \alpha \in \Gamma_d,$$

since $\left\langle w^{\frac{1}{2}} \right\rangle_{E_{\beta,J}}^2 \lesssim \left\langle w^{\frac{1}{2}} \right\rangle_J^2 \leq \langle w \rangle_J$. Applying the linear bound of the square function to $w^{-1} \mathbf{1}_{E_{\alpha,I}}$, again using trivial estimates, yields

$$(2.13) \quad \sum_{J \subseteq I} \sum_{\beta: E_{\beta,J} \subseteq E_{\alpha,I}} \left| \widehat{w^{-1}}(J, \beta) \right|^2 \langle w \rangle_{E_{\beta,J}} \lesssim [w]_{A_2}^2 w^{-1}(E_{\alpha,I}) \quad \forall I \in \mathcal{D}, \alpha \in \Gamma_d.$$

Because of the symmetry of the A_2 condition, we additionally have these estimates with the roles of w and w^{-1} interchanged. These estimates will all play a fundamental role when applying the modified Carleson Embedding Theorem.

3. LINEAR BOUND FOR HAAR MULTIPLIERS

We now turn to proving Theorem 1.2. Given a sequence $\sigma = \{\sigma_{I,\alpha}\}_{I \in \mathcal{D}, \alpha \in \Gamma_d}$ we define the Haar multiplier by

$$T_\sigma f \equiv \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \widehat{f}(I, \alpha) h_I^\alpha \quad \forall f \in L^2.$$

We must show that

$$\left\| Q_{T_\sigma, w}^{(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4)} \right\|_{L^2 \rightarrow L^2} \lesssim [w]_{A_2} \|\sigma\|_\infty$$

where the operators $Q_{T_\sigma, w}^{(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4)}$ are defined via (1.1) and the canonical decomposition of $M_{w^{\pm \frac{1}{2}}}$ into paraproducts is given in (2.3).

3.1. Estimating the Easy Terms. There are four easy terms that arise from (1.1). They are easy because the composition of the paraproducts reduce to classical paraproduct type operators. The terms are:

$$(3.1) \quad P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)};$$

$$(3.2) \quad P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} T_\sigma P_{\langle \widehat{w^{-\frac{1}{2}}} \rangle}^{(0,0)};$$

$$(3.3) \quad P_{\langle \widehat{w^{\frac{1}{2}}} \rangle}^{(0,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)};$$

$$(3.4) \quad P_{\langle \widehat{w^{\frac{1}{2}}} \rangle}^{(0,0)} T_\sigma P_{\langle \widehat{w^{-\frac{1}{2}}} \rangle}^{(0,0)}.$$

For these terms, we proceed by computing the norm of the operators in question by using duality. Key to this will be the application of the modified Carleson Embedding Theorem.

3.1.1. Estimating $P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)}$. Fix $\phi, \psi \in L^2$ and observe that

$$\begin{aligned} P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)} \phi &= P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) h_I^\alpha \\ &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \widehat{w^{\frac{1}{2}}}(I, \alpha) \widehat{w^{-\frac{1}{2}}}(I, \alpha) \langle \phi \rangle_{E_{\alpha,I}} h_{E_{\alpha,I}}^1. \end{aligned}$$

Then, we can calculate:

$$\begin{aligned}
 (3.5) \quad \left| \left\langle \widehat{P_{\frac{1}{2}}^{(1,0)}} T_{\sigma} \widehat{P_{\frac{1}{2}}^{(0,1)}} \phi, \psi \right\rangle_{L^2} \right| &= \left| \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \widehat{w^{\frac{1}{2}}}(I, \alpha) \widehat{w^{-\frac{1}{2}}}(I, \alpha) \langle \phi \rangle_{E_{\alpha,I}} \langle \psi \rangle_{E_{\alpha,I}} \right| \\
 &\leq \|\sigma\|_{\infty} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} |\widehat{w^{\frac{1}{2}}}(I, \alpha) \widehat{w^{-\frac{1}{2}}}(I, \alpha)| \langle \phi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
 &\quad \times \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} |\widehat{w^{\frac{1}{2}}}(I, \alpha) \widehat{w^{-\frac{1}{2}}}(I, \alpha)| \langle \psi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
 &\lesssim \|\sigma\|_{\infty} [w]_{A_2}^{\frac{1}{2}} \|\phi\|_{L^2} \|\psi\|_{L^2},
 \end{aligned}$$

where the last inequality follows from the Carleson Embedding Theorem. It applies here, since Cauchy-Schwarz gives:

$$\begin{aligned}
 \frac{1}{|E_{\alpha,I}|} \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \left| \widehat{w^{\frac{1}{2}}}(I, \beta) \widehat{w^{-\frac{1}{2}}}(I, \beta) \right| &\leq \frac{1}{|E_{\alpha,I}|} \left\| w^{\frac{1}{2}} \mathbf{1}_{E_{\alpha,I}} \right\|_{L^2} \left\| w^{-\frac{1}{2}} \mathbf{1}_{E_{\alpha,I}} \right\|_{L^2} \\
 &= (\langle w \rangle_{E_{\alpha,I}} \langle w^{-1} \rangle_{E_{\alpha,I}})^{\frac{1}{2}} \\
 (3.6) \quad &\lesssim [w]_{A_2}^{\frac{1}{2}}.
 \end{aligned}$$

Taking the supremum over all $\phi, \psi \in L^2$ and using duality gives

$$\left\| \widehat{P_{\frac{1}{2}}^{(1,0)}} T_{\sigma} \widehat{P_{\frac{1}{2}}^{(0,1)}} \right\|_{L^2 \rightarrow L^2} \lesssim \|\sigma\|_{\infty} [w]_{A_2}^{\frac{1}{2}} \leq \|\sigma\|_{\infty} [w]_{A_2}.$$

3.1.2. *Estimating $\widehat{P_{\frac{1}{2}}^{(1,0)}} T_{\sigma} \widehat{P_{\frac{1}{2}}^{(0,0)}}$ and $\widehat{P_{\frac{1}{2}}^{(0,0)}} T_{\sigma} \widehat{P_{\frac{1}{2}}^{(0,1)}}$.* As these two operators are symmetric, very similar arguments can be used to control both of them. Thus, we only provide details for the first operator. Observe that

$$\widehat{P_{\frac{1}{2}}^{(1,0)}} T_{\sigma} \widehat{P_{\frac{1}{2}}^{(0,0)}} \phi = \widehat{P_{\frac{1}{2}}^{(1,0)}} \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle w^{-\frac{1}{2}} \rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) h_I^{\alpha} = \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle w^{-\frac{1}{2}} \rangle_{E_{\alpha,I}} \widehat{w^{\frac{1}{2}}}(I, \alpha) \widehat{\phi}(I, \alpha) h_{E_{\alpha,I}}^1.$$

Fixing $\phi, \psi \in L^2$, we can calculate

$$\begin{aligned}
 (3.7) \quad \left| \left\langle \widehat{P_{\frac{1}{2}}^{(1,0)}} T_{\sigma} \widehat{P_{\frac{1}{2}}^{(0,0)}} \phi, \psi \right\rangle_{L^2} \right| &= \left| \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle w^{-\frac{1}{2}} \rangle_{E_{\alpha,I}} \widehat{w^{\frac{1}{2}}}(I, \alpha) \widehat{\phi}(I, \alpha) \langle \psi \rangle_{E_{\alpha,I}} \right| \\
 &\leq \|\sigma\|_{\infty} \|\phi\|_{L^2} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \widehat{w^{\frac{1}{2}}}(I, \alpha) \right|^2 \langle w^{-\frac{1}{2}} \rangle_{E_{\alpha,I}}^2 \langle \psi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
 &\lesssim \|\sigma\|_{\infty} [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2},
 \end{aligned}$$

where the last inequality follows from the Carleson Embedding Theorem and estimate (2.12). Again, taking the supremum over all $\phi, \psi \in L^2$ and using duality gives

$$\left\| \widehat{P_{\frac{1}{2}}^{(1,0)}} T_{\sigma} \widehat{P_{\frac{1}{2}}^{(0,0)}} \right\|_{L^2 \rightarrow L^2} \lesssim \|\sigma\|_{\infty} [w]_{A_2}.$$

3.1.3. *Estimating $P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} T_\sigma P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)}$.* Fixing $\phi, \psi \in L^2$, observe that

$$\begin{aligned} P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} T_\sigma P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)} \phi &= P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) h_I^\alpha \\ &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \left\langle w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) h_I^\alpha. \end{aligned}$$

This means we can calculate

$$\begin{aligned} (3.8) \quad \left| \left\langle P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} T_\sigma P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)} \phi, \psi \right\rangle_{L^2} \right| &= \left| \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \left\langle w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \widehat{\psi}(I, \alpha) \right| \\ &\lesssim \|\sigma\|_\infty \sup_{I \in \mathcal{D}} \left(\left\langle w^{\frac{1}{2}} \right\rangle_I \left\langle w^{-\frac{1}{2}} \right\rangle_I \right) \|\phi\|_{L^2} \|\psi\|_{L^2} \\ &\leq \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \|\phi\|_{L^2} \|\psi\|_{L^2}. \end{aligned}$$

Taking the supremum over all $\phi, \psi \in L^2$ and using duality gives the desired linear norm bound. This concludes the proof for the easy terms.

3.2. Estimating the Hard Terms. There are five remaining terms to be controlled. These include the four difficult terms:

$$(3.9) \quad \widehat{P_{\langle w^{\frac{1}{2}} \rangle}^{(0,1)}} T_\sigma \widehat{P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,1)}};$$

$$(3.10) \quad \widehat{P_{\langle w^{\frac{1}{2}} \rangle}^{(0,1)}} T_\sigma P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)};$$

$$(3.11) \quad P_{\langle w^{\frac{1}{2}} \rangle}^{(1,0)} T_\sigma \widehat{P_{\langle w^{-\frac{1}{2}} \rangle}^{(1,0)}};$$

$$(3.12) \quad P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} T_\sigma \widehat{P_{\langle w^{-\frac{1}{2}} \rangle}^{(1,0)}}.$$

To estimate terms (3.9) and (3.11) we will rely on disbalanced Haar functions adapted to the weights w and w^{-1} . For these terms, we also compute the norms using duality and frequent application of the modified Carleson Embedding Theorem. The proof of the estimates for these terms is carried out in subsection 3.2.1. Terms (3.10) and (3.12) will be handled via a similar method; their analysis appears in subsection 3.2.2.

The remaining term is the one for which T_σ can not be absorbed into one of the paraproducts. Namely, we need to control the following expression:

$$(3.13) \quad \widehat{P_{\langle w^{\frac{1}{2}} \rangle}^{(0,1)}} T_\sigma \widehat{P_{\langle w^{-\frac{1}{2}} \rangle}^{(1,0)}}.$$

To handle this term, we must rely on Theorem 1.1 and the computed linear bounds for the other eight paraproduct compositions. This leaves open the question of whether there is an independent proof of the linear bound for (3.13). This is discussed further in subsection 3.2.3.

3.2.1. *Estimating $P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)}$ and $P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)}$.* Similar arguments handle both terms and so, we restrict attention to the first one. Fix $\phi, \psi \in L^2$. Observe that basic manipulations and the product formula (2.2) for Haar coefficients give

$$\begin{aligned} \left\langle P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)} \phi, \psi \right\rangle_{L^2} &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) \sum_{\beta \in \Gamma_d} \sum_{J \in \mathcal{D}} \widehat{w^{\frac{1}{2}}}(J, \beta) \widehat{\psi}(J, \beta) \left\langle h_I^\alpha, h_{E_{\beta,J}}^1 \right\rangle_{L^2} \\ &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) \\ &\quad \times \left(\widehat{\psi w^{\frac{1}{2}}}(I, \alpha) - \widehat{\psi}(I, \alpha) \left\langle w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} - \widehat{w^{\frac{1}{2}}}(I, \alpha) \langle \psi \rangle_{E_{\alpha,I}} \right) \\ &\equiv T_1 + T_2 + T_3. \end{aligned}$$

We will show that each $|T_j| \lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}$. The bounds for T_2 and T_3 follow easily. First, observe that

$$\begin{aligned} |T_2| &\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \langle \phi \rangle_{E_{\alpha,I}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{\psi}(I, \alpha) \left\langle w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \right| \\ &\leq \|\sigma\|_\infty \|\psi\|_{L^2} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \widehat{w^{-\frac{1}{2}}}(I, \alpha) \right|^2 \left\langle w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}}^2 \langle \phi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\sigma\|_\infty [w]_{A_2} \|\psi\|_{L^2} \|\phi\|_{L^2}. \end{aligned}$$

The last inequality follows via the Carleson Embedding Theorem and the square function estimate (2.12). For T_3 , the computations are similarly straightforward:

$$\begin{aligned} |T_3| &\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{w^{\frac{1}{2}}}(I, \alpha) \langle \phi \rangle_{E_{\alpha,I}} \langle \psi \rangle_{E_{\alpha,I}} \right| \\ &\leq \|\sigma\|_\infty \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{w^{\frac{1}{2}}}(I, \alpha) \right| \langle \phi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{w^{\frac{1}{2}}}(I, \alpha) \right| \langle \psi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \|\phi\|_{L^2} \|\psi\|_{L^2}. \end{aligned}$$

Here, the last inequality follows from two applications of the Carleson Embedding Theorem using the estimate given in (3.6). Estimating T_1 requires the use of disbalanced Haar functions. We expand the Haar functions in the sum using two disbalanced systems, one

associated to w and one associated to w^{-1} , as follows:

$$\begin{aligned}
(3.14) \quad T_1 &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{\psi w^{\frac{1}{2}}}(I, \alpha) \\
&= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} \left\langle w^{-\frac{1}{2}}, C_I(w^{-1}, \alpha) h_I^{w^{-1}, \alpha} + D_I(w^{-1}, \alpha) h_{E_{\alpha,I}}^1 \right\rangle_{L^2} \\
&\quad \times \left\langle \psi w^{\frac{1}{2}}, C_I(w, \alpha) h_I^{w, \alpha} + D_I(w, \alpha) h_{E_{\alpha,I}}^1 \right\rangle_{L^2} \\
&= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} C_I(w^{-1}, \alpha) C_I(w, \alpha) \left\langle w^{-\frac{1}{2}}, h_I^{w^{-1}, \alpha} \right\rangle_{L^2} \left\langle \psi w^{\frac{1}{2}}, h_I^{w, \alpha} \right\rangle_{L^2} \\
&\quad + \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} C_I(w^{-1}, \alpha) D_I(w, \alpha) \left\langle w^{-\frac{1}{2}}, h_I^{w^{-1}, \alpha} \right\rangle_{L^2} \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \\
&\quad + \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} D_I(w^{-1}, \alpha) C_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \left\langle \psi w^{\frac{1}{2}}, h_I^{w, \alpha} \right\rangle_{L^2} \\
&\quad + \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \langle \phi \rangle_{E_{\alpha,I}} D_I(w^{-1}, \alpha) D_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \\
&\equiv S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

Now, we show each $|S_j| \lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}$, which gives the bound for T_1 . Observe that by (2.6),

$$\begin{aligned}
|S_1| &\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \left\langle \phi \right\rangle_{E_{\alpha,I}} C_I(w^{-1}, \alpha) C_I(w, \alpha) \left\langle w^{-\frac{1}{2}}, h_I^{w^{-1}, \alpha} \right\rangle_{L^2} \left\langle \psi w^{-\frac{1}{2}}, h_I^{w, \alpha} \right\rangle_{L^2(w)} \right| \\
&\lesssim \|\sigma\|_\infty \left\| \psi w^{-\frac{1}{2}} \right\|_{L^2(w)} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left\langle w^{-1} \right\rangle_I \left\langle w \right\rangle_I \left\langle w^{-\frac{1}{2}}, h_I^{w^{-1}, \alpha} \right\rangle_{L^2}^2 \left\langle \phi \right\rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \|\psi\|_{L^2} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left\langle w^{-\frac{1}{2}}, h_I^{w^{-1}, \alpha} \right\rangle_{L^2}^2 \left\langle \phi \right\rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \|\psi\|_{L^2} \|\phi\|_{L^2},
\end{aligned}$$

where the last inequality followed via the Carleson Embedding Theorem using the estimate

$$\sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \left\langle w^{-\frac{1}{2}}, h_J^{w^{-1}, \beta} \right\rangle_{L^2}^2 = \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \left\langle w^{\frac{1}{2}}, h_J^{w^{-1}, \beta} \right\rangle_{L^2(w^{-1})}^2 \leq \left\| w^{\frac{1}{2}} \mathbf{1}_{E_{\alpha,I}} \right\|_{L^2(w^{-1})}^2 = |E_{\alpha,I}|.$$

The calculation for S_2 is also straightforward:

$$\begin{aligned}
|S_2| &\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \left\langle \phi \right\rangle_{E_{\alpha,I}} C_I(w^{-1}, \alpha) D_I(w, \alpha) \left\langle w^{-\frac{1}{2}}, h_I^{w^{-1}, \alpha} \right\rangle_{L^2} \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \right| \\
&\lesssim \|\sigma\|_\infty \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left\langle w^{-\frac{1}{2}}, h_I^{w^{-1}, \alpha} \right\rangle_{L^2}^2 \left\langle \phi \right\rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left\langle w^{-1} \right\rangle_{E_{\alpha,I}} \frac{|\widehat{w}(I, \alpha)|^2}{\left\langle w \right\rangle_{E_{\alpha,I}}^2} \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2},
\end{aligned}$$

where we use the Carleson Embedding Theorem twice. The application for the ϕ term follows as in the estimate for S_1 , while the application for ψ follows from the square function estimate (2.13). Similarly, for S_3 , we can calculate

$$\begin{aligned}
|S_3| &\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \langle \phi \rangle_{E_{\alpha,I}} D_I(w^{-1}, \alpha) C_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \left\langle \psi w^{\frac{1}{2}}, h_I^{w,\alpha} \right\rangle_{L^2} \right| \\
&\lesssim \|\sigma\|_\infty \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \frac{|\widehat{w^{-1}}(I, \alpha)|^2}{\langle w^{-1} \rangle_{E_{\alpha,I}}^2} \langle \phi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \langle w \rangle_I \left\langle w^{-\frac{1}{2}} \right\rangle_I^2 \left\langle \psi w^{-\frac{1}{2}}, h_I^{w,\alpha} \right\rangle_{L^2(w)}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \left\| \psi w^{-\frac{1}{2}} \right\|_{L^2(w)} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \frac{|\widehat{w^{-1}}(I, \alpha)|^2}{\langle w^{-1} \rangle_{E_{\alpha,I}}^2} \langle \phi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2},
\end{aligned}$$

where the last inequality follows from the Carleson Embedding Theorem. It applies here since:

$$(3.15) \quad \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \frac{|\widehat{w^{-1}}(J, \beta)|^2}{\langle w^{-1} \rangle_{E_{\beta,J}}^2} \lesssim [w]_{A_2} |E_{\alpha,I}| \quad \forall \alpha \in \Gamma_d \quad \forall I \in \mathcal{D}.$$

To see that (3.15) holds, it is then a simple application of Cauchy-Schwarz and the following estimates:

$$(3.16) \quad \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \frac{|\widehat{w^{-1}}(J, \beta)|^2}{\langle w^{-1} \rangle_{E_{\beta,J}}^2} \lesssim [w]_{A_2} w^{-1}(E_{\alpha,I}) \quad \forall \alpha \in \Gamma_d \quad \forall I \in \mathcal{D};$$

$$(3.17) \quad \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \frac{|\widehat{w^{-1}}(J, \beta)|^2}{\langle w^{-1} \rangle_{E_{\beta,J}}^3} \lesssim w(E_{\alpha,I}) \quad \forall \alpha \in \Gamma_d \quad \forall I \in \mathcal{D}.$$

The proofs of (3.16) and (3.17) can be found in [1, Proposition 4.9, Equation (4.17)] and [1, Proposition 4.7, Equation (4.11)] respectively.

Lastly, the estimate for S_4 is computed as follows:

$$\begin{aligned}
|S_4| &\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \langle \phi \rangle_{E_{\alpha,I}} D_I(w^{-1}, \alpha) D_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \right| \\
&\leq \|\sigma\|_\infty \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} |\widehat{w}(I, \alpha) \widehat{w^{-1}}(I, \alpha)| \langle \phi \rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \frac{|\widehat{w}(I, \alpha) \widehat{w^{-1}}(I, \alpha)|}{\langle w \rangle_{E_{\alpha,I}}^2 \langle w^{-1} \rangle_{E_{\alpha,I}}} \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
&= \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2},
\end{aligned}$$

where the Carleson Embedding Theorem is used twice. The application for the ϕ term uses

$$(3.18) \quad \sum_{J \subset I} \sum_{\beta: E_{\beta,J} \subset E_{\alpha,I}} \left| \widehat{w}(J, \beta) \widehat{w^{-1}}(J, \beta) \right| \lesssim [w]_{A_2} |E_{\alpha,I}| \quad \forall \alpha \in \Gamma_d \quad \forall I \in \mathcal{D}.$$

As stated, the proof of this is found in [1, Equation (6.3)]. The one-dimensional version is established in [9, Lemma 4.7]. The application for the ψ term uses

$$(3.19) \quad \sum_{J \subset I} \sum_{\beta: E_\beta, J \subset E_{\alpha, I}} \frac{|\widehat{w}(J, \beta) \widehat{w^{-1}}(J, \beta)|}{\langle w^{-1} \rangle_{E_{\beta, J}}} \lesssim [w]_{A_2} w(E_{\alpha, I}) \quad \forall \alpha \in \Gamma_d \quad \forall I \in \mathcal{D}.$$

As stated, the proof of this is found in [1, Equation (6.4)].

This concludes the proof of the estimates for T_1, T_2, T_3 . By taking the supremum over $\phi, \psi \in L^2$ and using duality, we conclude that

$$\left\| P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)} \right\|_{L^2 \rightarrow L^2} \lesssim \|\sigma\|_\infty [w]_{A_2}.$$

3.2.2. Estimating $P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,0)}$ and $P_{\widehat{w^{\frac{1}{2}}}}^{(0,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)}$. We only discuss the first operator, as the estimates for the second one follow via similar arguments. Fix $\phi, \psi \in L^2$. We first simplify using basic manipulations and the product formula for Haar coefficients, (2.2), as follows:

$$\begin{aligned} \left\langle P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,0)} \phi, \psi \right\rangle_{L^2} &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I, \alpha} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha, I}} \widehat{\phi}(I, \alpha) \sum_{\beta \in \Gamma_d} \sum_{J \in \mathcal{D}} \widehat{w^{\frac{1}{2}}}(J, \beta) \widehat{\psi}(J, \beta) \left\langle h_I^\alpha, h_{E_{\beta, J}}^1 \right\rangle_{L^2} \\ &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I, \alpha} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha, I}} \widehat{\phi}(I, \alpha) \\ &\quad \times \left(\widehat{\psi w^{\frac{1}{2}}}(I, \alpha) - \widehat{\psi}(I, \alpha) \left\langle w^{\frac{1}{2}} \right\rangle_{E_{\alpha, I}} - \widehat{w^{\frac{1}{2}}}(I, \alpha) \langle \psi \rangle_{E_{\alpha, I}} \right) \\ &\equiv T_1 + T_2 + T_3. \end{aligned}$$

As in the previous case, we show that each $|T_j| \lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}$. The estimates for T_2 and T_3 follow easily. Observe that

$$\begin{aligned} |T_2| &\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha, I}} \left\langle w^{\frac{1}{2}} \right\rangle_{E_{\alpha, I}} \widehat{\phi}(I, \alpha) \widehat{\psi}(I, \alpha) \right| \\ &\lesssim \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \widehat{\phi}(I, \alpha) \widehat{\psi}(I, \alpha) \right| \\ &\leq \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \|\phi\|_{L^2} \|\psi\|_{L^2}, \end{aligned}$$

and similarly,

$$\begin{aligned} |T_3| &\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha, I}} \langle \psi \rangle_{E_{\alpha, I}} \widehat{\phi}(I, \alpha) \widehat{w^{\frac{1}{2}}}(I, \alpha) \right| \\ &\leq \|\sigma\|_\infty \|\phi\|_{L^2} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha, I}}^2 \left| \widehat{w^{\frac{1}{2}}}(I, \alpha) \right|^2 \langle \psi \rangle_{E_{\alpha, I}}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}, \end{aligned}$$

where the last inequality followed via an application of the Carleson Embedding Theorem using (2.12).

To estimate T_1 , we rewrite the term $\left\langle \psi w^{\frac{1}{2}}, h_I^\alpha \right\rangle_{L^2}$ using disbalanced Haar functions adapted to w as follows:

$$\begin{aligned}
(3.20) \quad T_1 &= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \widehat{\psi w^{\frac{1}{2}}}(I, \alpha) \\
&= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \left\langle \psi w^{\frac{1}{2}}, C_I(w, \alpha) h_I^{w,\alpha} + D_I(w, \alpha) h_{E_{\alpha,I}}^1 \right\rangle_{L^2} \\
&= \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} C_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \left\langle \psi w^{\frac{1}{2}}, h_I^{w,\alpha} \right\rangle_{L^2} \\
&\quad + \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} D_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \\
&= S_1 + S_2.
\end{aligned}$$

Now, we show each $|S_j| \lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}$, which will give the estimate for T_1 . First, consider S_1 :

$$\begin{aligned}
|S_1| &= \left| \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} C_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \left\langle \psi w^{\frac{1}{2}}, h_I^{w,\alpha} \right\rangle_{L^2} \right| \\
&\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| C_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \left\langle \psi w^{\frac{1}{2}}, h_I^{w,\alpha} \right\rangle_{L^2} \right| \\
&\lesssim \|\sigma\|_\infty \|\phi\|_{L^2} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \langle w \rangle_I \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}}^2 \left\langle \psi w^{-\frac{1}{2}}, h_I^{w,\alpha} \right\rangle_{L^2(w)}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \|\phi\|_{L^2} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left\langle \psi w^{-\frac{1}{2}}, h_I^{w,\alpha} \right\rangle_{L^2(w)}^2 \right)^{\frac{1}{2}} \\
&\leq \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \|\phi\|_{L^2} \left\| \psi w^{-\frac{1}{2}} \right\|_{L^2(w)} \\
&= \|\sigma\|_\infty [w]_{A_2}^{\frac{1}{2}} \|\phi\|_{L^2} \|\psi\|_{L^2}.
\end{aligned}$$

Above we used (2.6) coupled with the A_2 condition. Lastly, we estimate S_2 as follows:

$$\begin{aligned}
|S_2| &\leq \left| \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \sigma_{I,\alpha} D_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \right| \\
&\leq \|\sigma\|_\infty \sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \left| D_I(w, \alpha) \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \widehat{\phi}(I, \alpha) \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}} \right| \\
&\leq \|\sigma\|_\infty \|\phi\|_{L^2} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \frac{|\widehat{w}(I, \alpha)|^2}{\langle w \rangle_{E_{\alpha,I}}^2} \left\langle w^{-\frac{1}{2}} \right\rangle_{E_{\alpha,I}}^2 \left\langle \psi w^{\frac{1}{2}} \right\rangle_{E_{\alpha,I}}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2},
\end{aligned}$$

where the third inequality follows via an application of the Carleson Embedding Theorem using estimate (2.13). This establishes that each $|T_j| \lesssim \|\sigma\|_\infty [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}$. Since $\phi, \psi \in$

L^2 were arbitrary, we can use duality to conclude

$$\left\| P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)} \right\|_{L^2 \rightarrow L^2} \lesssim \|\sigma\|_\infty [w]_{A_2},$$

which finishes the proof of these terms.

3.2.3. *Estimating $P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)}$.* To obtain estimates on the final term $P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)}$ we simply use (1.1) to observe that

$$\begin{aligned} \left\| P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)} \phi \right\|_{L^2} &\leq \left\| M_{\widehat{w^{\frac{1}{2}}}} T_\sigma M_{\widehat{w^{-\frac{1}{2}}}} \phi \right\|_{L^2} + \left\| P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)} \phi \right\|_{L^2} + \left\| P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} T_\sigma P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)} \phi \right\|_{L^2} \\ &\quad + \left\| P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)} \phi \right\|_{L^2} + \left\| P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} T_\sigma P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)} \phi \right\|_{L^2} + \left\| P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(0,1)} \phi \right\|_{L^2} \\ &\quad + \left\| P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\langle w^{-\frac{1}{2}} \rangle}^{(0,0)} \phi \right\|_{L^2} + \left\| P_{\widehat{w^{\frac{1}{2}}}}^{(1,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)} \phi \right\|_{L^2} + \left\| P_{\langle w^{\frac{1}{2}} \rangle}^{(0,0)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)} \phi \right\|_{L^2} \\ &\lesssim [w]_{A_2} \|\phi\|_{L^2}, \end{aligned}$$

using Theorem 1.1 and our previous computations. This proof strategy motivates the open question:

Question 3.1. Is there a proof of the linear bound for the final paraproduct composition $P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)}$ that does not rely on the linear bound of T_σ on $L^2(w)$?

To be precise, fix $\phi, \psi \in L^2$. Then, the term of interest is

$$\begin{aligned} \left\langle P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma P_{\widehat{w^{-\frac{1}{2}}}}^{(1,0)} \phi, \psi \right\rangle_{L^2} &= \left\langle P_{\widehat{w^{\frac{1}{2}}}}^{(0,1)} T_\sigma \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{\phi}(I, \alpha) h_{E_{\alpha, I}}^1 \right), \psi \right\rangle_{L^2} \\ &= \sum_{\alpha, \beta \in \Gamma_d} \sum_{I, J \in \mathcal{D}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{\phi}(I, \alpha) \widehat{w^{\frac{1}{2}}}(J, \beta) \widehat{\psi}(J, \beta) \left\langle T_\sigma h_{E_{\alpha, I}}^1, h_{E_{\beta, J}}^1 \right\rangle_{L^2} \\ &= \sum_{\alpha, \beta, \gamma \in \Gamma_d} \sum_{I, J, K \in \mathcal{D}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{\phi}(I, \alpha) \widehat{w^{\frac{1}{2}}}(J, \beta) \widehat{\psi}(J, \beta) \sigma_{K, \gamma} \widehat{h_{E_{\alpha, I}}^1}(K, \gamma) \widehat{h_{E_{\beta, J}}^1}(K, \gamma) \\ &= \sum_{\gamma \in \Gamma_d} \sum_{K \in \mathcal{D}} \sigma_{K, \gamma} \left(\sum_{\alpha \in \Gamma_d} \sum_{I \in \mathcal{D}} \widehat{w^{-\frac{1}{2}}}(I, \alpha) \widehat{\phi}(I, \alpha) \widehat{h_{E_{\alpha, I}}^1}(K, \gamma) \right) \\ &\quad \times \left(\sum_{\beta \in \Gamma_d} \sum_{J \in \mathcal{D}} \widehat{w^{\frac{1}{2}}}(J, \beta) \widehat{\psi}(J, \beta) \widehat{h_{E_{\beta, J}}^1}(K, \gamma) \right). \end{aligned}$$

Currently, our tools seem unequal to the task of bounding this term without recourse to the bound for T_σ . However, given such arguments, one would also obtain a new proof of the linear bound for T_σ on $L^2(w)$.

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